Some Non-Trivial Kazhdan-Lusztig Coefficients of an Affine Weyl Group of Type $\tilde{A}_n$

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ABSTRACT. In this paper we show that the leading coefficient $\mu(y,w)$ of some Kazhdan-Lusztig polynomials $P_{y,w}$ with y,w in an affine Weyl group of type $\tilde{A}_n$ is $n+2$. This fact has some consequences on the dimension of first extension groups of finite groups of Lie type with irreducible coefficients.

Given two elements $y \le w$ in a Coxeter group (W,S) (S the set of simple reflections), we have a Kazhdan-Lusztig polynomial $P_{y,w}$ in an indeterminate q. If $y < w$, the degree of $P_{y,w}$ is less than or equal to $\frac{1}{2}(l(w)-l(y)-1)$. Particularly interesting is the coefficient $\mu(y,w)$ of the term $q^{\frac{1}{2}(l(w)-l(y)-1)}$ in $P_{y,w}$, since it plays a key role in understanding Kazhdan-Lusztig polynomials and in a recursive formula for them. Moreover, this "leading" coefficient (it can be zero) is important in representation theory and in understanding cohomology and first extension groups for irreducible modules of algebraic groups and of finite groups of Lie type.

However, it is in general hard to compute the leading coefficient. In [**L6**] Lusztig computes the leading coefficient for some Kazhdan-Lusztig polynomials of an affine Weyl group of type $\tilde{B}_2$, more are computed in [**W**]. In [**S**] for an affine Weyl group of type $\tilde{A}_5$, some non-trivial leading coefficients are worked out. McLarnan and Warrington have shown that $\mu(y,w)$ can be greater than 1 for symmetric groups, see

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[**MW**]. In [**X3**], Xi shows that if $a(y) < a(w)$, then $\mu(y,w) \leq 1$ when W is a symmetric group or an affine Weyl group of type $\tilde{A}_n$.

In this paper we show that the leading coefficient $\mu(y,w)$ of some Kazhdan-Lusztig polynomials $P_{y,w}$ with y,w in an affine Weyl group of type $\tilde{A}_n$ is $n+2$ (see Theorem 3.3). There is a well-known identification [**A1**] of this coefficient with dimensions of first extension groups for irreducible modules of the underlying algebraic group, which here is $SL_{n+1}(\bar{\mathbb{F}}_p)$, in the presence of the Lusztig conjecture (known to hold for p very large [**AJS**]). Thus, our results show the dimensions of these first extension groups can be arbitrarily large as n becomes large. Taken together with [**CPS2**], this implies that the corresponding first extension groups for the finite groups $SL_{n+1}(\mathbb{F}_q)$, q a sufficiently large power of (a sufficiently large) prime p, must also have unbounded dimensions. In particular, a well-known conjecture of Robert Guralnick [**G**], that there exists a universal constant bound on dimensions of the first cohomology groups of finite groups (with faithful absolutely irreducible modules as coefficients), cannot be extended to first extension groups.

In Section 5 we give a representation-theoretic approach to (a variation on) Theorem 3.3. It does not yield the same precise calculation[1], but applies to more weights; see Remark 5.3(c). More importantly, it yields an independent confirmation of the fact demonstrated by Theorem 3.3, that the coefficients $\mu(y,w)$, and the dimensions of first extension groups which correspond to them, can go to infinity with n.

1. Springer's formula

In this section we recall some basic facts and a formula of Springer for the leading coefficient $\mu(y,w)$.

1.1. Let G be a connected, simply connected reductive algebraic group over the field $\mathbf{C}$ of complex numbers and T a maximal torus of G. Let $N_G(T)$ be the normalizer of T in G. Then $W_0 = N_G(T)/T$ is

[1] H. Andersen has recently provided a way to obtain precise formulas, as in Theorem 3.3, from the representation-theoretic approach of Section 5 (but still using the Coxeter group lemma, Lemma 3.4), by using some homological results of [**AJ**]. We sketch Andersen's argument in Remark 5.3 (d).

a Weyl group, which acts on the character group $X = \mathrm{Hom}(T, \mathbf{C}^*)$ of T. The semi-direct product $W_0 \ltimes X$ is called an extended affine Weyl group, denoted by W. It contains the affine Weyl group W_a, the semi-direct product of W_0 and the root lattice. We shall denote by S the set of simple reflections of W. We shall denote the length function of W by l and use $\leq$ for the Bruhat order on W. We refer to subsection 2.1 for a formula of the length function, see also subsections 1.1 and 1.2 in [**L5**] or section 1.1 in [**X2**] for the length function and Bruhat order. See also [**L2**], where Lusztig explains how to carry over notions from [**KL**], including Kazhdan-Lusztig polynomials, to (W, S) and a Hecke algebra for it.

Let H be the Hecke algebra of (W, S) over $\mathcal{A} = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ (q an indeterminate) with parameter q. Let $\{T_w\}_{w \in W}$ be its standard basis. Let $C_w = q^{-\frac{l(w)}{2}} \sum_{y \leq w} P_{y,w} T_y$, $w \in W$ be the Kazhdan-Lusztig basis of H, where $P_{y,w}$ are the Kazhdan-Lusztig polynomials. The degree of $P_{y,w}$ is less than or equal to $\frac{1}{2}(l(w) - l(y) - 1)$ if $y < w$. We write $P_{y,w} = \mu(y, w) q^{\frac{1}{2}(l(w)-l(y)-1)}$+lower degree terms. The coefficient $\mu(y, w)$ is very interesting, this can be seen even from the recursive formula (see [**KL**]) for Kazhdan-Lusztig polynomials. We shall call $\mu(y, w)$ the *Kazhdan-Lusztig coefficient* of $P_{y,w}$. The extended and usual (non-extended) affine Weyl groups have essentially the same Kazhdan-Lusztig polynomials. For more details about Hecke algebras of extended affine Weyl groups, we refer to Section 4 in [**L2**], or Subsection 1.2 in [**L5**], or Sections 1.1 and 1.6 in [**X2**].

1.2. Write

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z, \qquad h_{x,y,z} \in \mathcal{A} = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

Following Lusztig ([**L3**]), we define

$$a(z) = \min\{i \in \mathbf{N} \mid q^{-\frac{i}{2}} h_{x,y,z} \in \mathbf{Z}[q^{-\frac{1}{2}}] \text{ for all } x, y \in W\}.$$

If for any i, $q^{-\frac{i}{2}} h_{x,y,z} \notin \mathbf{Z}[q^{-\frac{1}{2}}]$ for some $x, y \in W$, we set $a(z) = \infty$. Then $a(w) \leq l(w_0)$ for any $w \in W$, where w_0 is the longest element of W_0 (see [**L3**]]).

Following Lusztig and Springer, we define $\delta_{x,y,z}$ and $\gamma_{x,y,z}$ by the following formula,

$$h_{x,y,z} = \gamma_{x,y,z} q^{\frac{a(z)}{2}} + \delta_{x,y,z} q^{\frac{a(z)-1}{2}} + \text{ lower degree terms.}$$

Springer showed that $l(z) \geq a(z)$ (see [**L4**]). Let $\delta(z)$ be the degree of $P_{e,z}$, where e is the neutral element of W. Then actually one has $l(z) - a(z) - 2\delta(z) \geq 0$ (see [**L4**]). Set

$$\mathcal{D}_i = \{z \in W \mid l(z) - a(z) - 2\delta(z) = i\}.$$

The number $\pi(z)$ is defined by $P_{e,z} = \pi(z) q^{\delta(z)} +$ lower degree terms.

The elements of $\mathcal{D}_0$ are involutions, called distinguished involutions of (W, S) (see [**L4**]).

1.3. Assume that (W, S) is an extended affine Weyl group or a Weyl group. The following formula is due to Springer [**Sp**](see [**X2**] for a sketchy proof),

$$\begin{aligned} \mu(y,x) &= \sum_{d \in \mathcal{D}_0} \delta_{y^{-1},x,d} + \sum_{f \in \mathcal{D}_1} \gamma_{y^{-1},x,f} \pi(f) \\ &= \sum_{d \in \mathcal{D}_0} \delta_{y,x^{-1},d} + \sum_{f \in \mathcal{D}_1} \gamma_{y,x^{-1},f} \pi(f). \end{aligned}$$

1.4. We refer to [**KL**] for the definition of the preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$, $\underset{LR}{\leq}$ and of the equivalence relations $\underset{L}{\sim}$, $\underset{R}{\sim}$, $\underset{LR}{\sim}$ on W. The corresponding equivalence classes are called left cells, right cells, two-sided cells of W, respectively. The preorder $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}$; $\underset{LR}{\leq}$) induces a partial order on the set of left (resp. right; two-sided) cells of W, denoted again by $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}$; $\underset{LR}{\leq}$). For a Weyl group or an extended affine Weyl group, Springer showed [Sp] the following results (a) and (b) (see [**X3**])

(a) Assume that $\mu(y,w)$ or $\mu(w,y)$ is nonzero, then $y \underset{L}{\leq} w$ and $y \underset{R}{\leq} w$ if $a(y) < a(w)$, and $y \underset{L}{\sim} w$ or $y \underset{R}{\sim} w$ if $a(y) = a(w)$.

(b) If $\delta_{x,y,z} \neq 0$, then $z \underset{L}{\sim} y$ or $z \underset{R}{\sim} x$. (Note that $h_{x,y,z} \neq 0$ implies that $a(z) \geq a(x)$ and $a(z) \geq a(y)$, see [**L3**].)

For $w \in W$, set $L(w) = \{s \in S \mid sw \leq w\}$, $R(w) = \{s \in S \mid ws \leq w\}$. Then we have (see [**KL**])

(c) $R(w) \subseteq R(y)$ if $y \underset{L}{\leq} w$. In particular, $R(w) = R(y)$ if $y \underset{L}{\sim} w$;

(d) $L(w) \subseteq L(y)$ if $y \underset{R}{\leq} w$. In particular, $L(w) = L(y)$ if $y \underset{R}{\sim} w$.

2. The lowest two-sided cell

In this section we collect some facts about the lowest two-sided cell of W.

2.1. Let w_0 be the longest element of W_0. Let

$$\Gamma_0 = \{ww_0 \,|\, w \in W, \quad l(ww_0) = l(w) + l(w_0)\}.$$

Then Γ_0 is a left cell (see [**L3**]). The Kazhdan-Lusztig polynomials $P_{y,w}$ for y, w in Γ_0 play a key role in Lusztig's conjectures on irreducible characters of algebraic groups, of quantum groups at roots of unity and of affine Lie algebras. In this paper we are interested in the Kazhdan-Lusztig coefficient of $P_{y,w}$ for y, w in Γ_0.

It is known (see [**Sh1**]) that $c_0 = \{w \in W | a(w) = l(w_0)\}$ is a two-sided cell and contains Γ_0. In fact, c_0 is the lowest two-sided cell with respect to the partial order $\underset{LR}{\leq}$ on the set of two-sided cells of W and c_0 has $|W_0|$ left cells (see [**Sh2**]).

Let R^+ (resp. R^-, Δ) be the set of positive (resp. negative, simple) roots of the root system R of W_0. Then the length of xw ($w \in W_0,\ x \in X$) is given by the formula (see [**IM**])

$$l(xw) = \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^-}} |\langle x, \alpha^\vee \rangle + 1| + \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^+}} |\langle x, \alpha^\vee \rangle|.$$

Let $X^+ = \{x \in X | l(xw_0) = l(x) + l(w_0)\}$ be the set of dominant weights of X. For each simple root α we denote by s_α the corresponding simple reflection in W_0 and x_α the corresponding fundamental weight. Then we have $s_\alpha(y) = y - \langle y, \alpha^\vee \rangle \alpha$ for any $y \in X$ and $\langle x_\alpha, \beta^\vee \rangle = \delta_{\alpha\beta}$ for any simple roots $\alpha,\ \beta$. For each $w \in W_0$, we set

$$d_w = w \prod_{\substack{\alpha \in \Delta \\ w(\alpha) \in R^-}} x_\alpha.$$

Then

$$c_0 = \{d_w x w_0 d_u^{-1} | w, u \in W_0, x \in X^+\}.$$

Moreover, the set $c'_{0,w} = \{d_w x w_0 d_u^{-1} | u \in W_0, x \in X^+\}$ is a right cell of W and $c_{0,u} = \{d_w x w_0 d_u^{-1} | w \in W_0, x \in X^+\}$ is a left cell of W. The distinguished involutions of c_0 are $d_w w_0 d_w^{-1}$, $w \in W_0$. The set $\{d_w \mid w \in W_0\}$ can also be described as $\{z \in W \mid z w_0 \in c_0$ but $z w_0 s \notin c_0$ for any $s \in W_0 - \{e\}\}$. See [**Sh2**].

2.2. For $x \in X^+$ let $V(x)$ be a rational irreducible G-module of highest weight x and let S_x be the corresponding element defined in [**L2**]. Then S_x, $x \in X^+$, form an $\mathcal{A}$-basis of the center of H. For $w \in W_0$ we define

$$E_{d_w} = q^{-\frac{l(d_w)}{2}} \sum_{\substack{y \le d_w \\ l(yw_0)=l(y)+l(w_0)}} P_{yw_0, d_w w_0} T_y$$

and

$$F_{d_w} = q^{-\frac{l(d_w)}{2}} \sum_{\substack{y \le d_w \\ l(yw_0)=l(y)+l(w_0)}} P_{yw_0, d_w w_0} T_{y^{-1}}.$$

Then we have (see [**X1**, Corollary 2.11] and [**L2**, Proposition 8.6])

(a) $E_{d_w} S_x C_{w_0} F_{d_u} = C_{d_w x w_0 d_u^{-1}}$ for any $w, u \in W$ and $x \in X^+$.

(b) $S_x S_y = \sum_{z \in X^+} m_{x,y,z} S_z$ for any $x, y \in X^+$. Here $m_{x,y,z}$ is defined to be the multiplicity of $V(z)$ in the tensor product $V(x) \otimes V(y)$.

Using (a), (b) and 1.3 we get

(c) Let $w, w', u \in W_0$, $y, z \in X^+$ and let $d_w y w_0 d_u^{-1}$, $d_{w'} z w_0 d_u^{-1}$ be elements of the left cell $c_{0,u}$. Then $\mu(d_w y w_0 d_u^{-1}, d_{w'} z w_0 d_u^{-1}) = \mu(d_w y w_0, d_{w'} z w_0) = \mu(d_w z^* w_0, d_{w'} y^* w_0)$, here $y^* = w_0 y^{-1} w_0, z^* = w_0 z^{-1} w_0$. (We set $\mu(x, v) = \mu(v, x)$ if $v \le x$.)

We give some explanation for (c). By (a) we have

$$C_{d_u w_0 y^{-1} d_w^{-1}} C_{d_{w'} z w_0 d_u^{-1}} = E_{d_u} C_{w_0 y^{-1} d_w^{-1}} C_{d_{w'} z w_0} F_{d_u}.$$

Note that

$$C_{w_0 y^{-1} d_w^{-1}} C_{d_{w'} z w_0} = \sum_{x \in X^+} h_{w_0 y^{-1} d_w^{-1}, d_{w'} z w_0, x w_0} C_{x w_0}.$$

Using (a) we then get

$$C_{d_u w_0 y^{-1} d_w^{-1}} C_{d_{w'} z w_0 d_u^{-1}} = \sum_{x \in X^+} h_{w_0 y^{-1} d_w^{-1}, d_{w'} z w_0, x w_0} C_{d_u x w_0 d_u^{-1}}.$$

So we have

$$h_{d_u w_0 y^{-1} d_w^{-1}, d_{w'} z w_0 d_u^{-1}, d_u x w_0 d_u^{-1}} = h_{w_0 y^{-1} d_w^{-1}, d_{w'} z w_0, x w_0}.$$

This implies

$$\gamma_{d_u w_0 y^{-1} d_w^{-1}, d_{w'} z w_0 d_u^{-1}, d_u x w_0 d_u^{-1}} = \gamma_{w_0 y^{-1} d_w^{-1}, d_{w'} z w_0, x w_0},$$

$$\delta_{d_u w_0 y^{-1} d_w^{-1}, d_{w'} z w_0 d_u^{-1}, d_u x w_0 d_u^{-1}} = \delta_{w_0 y^{-1} d_w^{-1}, d_{w'} z w_0, x w_0}.$$

If $w \neq w'$, then $d_w y w_0 d_u^{-1}, d_{w'} z w_0 d_u^{-1}$ are in different right cells and $d_w y w_0, d_{w'} z w_0$ are in different right cells. So for any $x \in X^+$ we have

$$\gamma_{d_u w_0 y^{-1} d_w^{-1}, d_{w'} z w_0 d_u^{-1}, d_u x w_0 d_u^{-1}} = \gamma_{w_0 y^{-1} d_w^{-1}, d_{w'} z w_0, x w_0} = 0.$$

By the formula of Springer in 1.3 and the above formula for δ, we get

$$\mu(d_w y w_0 d_u^{-1}, d_{w'} z w_0 d_u^{-1}) = \delta_{d_u w_0 y^{-1} d_w^{-1}, d_{w'} z w_0 d_u^{-1}, d_u w_0 d_u^{-1}}$$

$$= \delta_{w_0 y^{-1} d_w^{-1}, d_{w'} z w_0, w_0} = \mu(d_w y w_0, d_{w'} z w_0).$$

If $w = w'$ and $d_w y w_0 d_u^{-1} \leq d_{w'} z w_0 d_u^{-1}$, then $y \leq z$. So $y^{-1} z$ is in the root lattice. Thus $l(z) - l(y)$ is even since $l(y^{-1} z)$ is even. Therefore the values of μ in (c) are all equal to 0 in this case. We have explained the first equality for μ in (c)

By (a) we get

$$C_{w_0 y^{-1} d_w^{-1}} C_{d_{w'} z w_0} = S_{y^*} S_z C_{w_0 d_w^{-1}} C_{d_{w'} w_0},$$

$$C_{w_0 (z^*)^{-1} d_w^{-1}} C_{d_{w'} y^* w_0} = S_{(z^*)^*} S_{y^*} C_{w_0 d_w^{-1}} C_{d_{w'} w_0}.$$

Since $(z^*)^* = z$, using the formula of Springer in 1.3 we see that the second equality in (c) is true.

3. Main results

Now we can state our main results.

Theorem 3.1. *There is a positive integer B such that $\mu(y, w) \leq B$ for all $y, w \in c_0$. In other words, the Kazhdan-Lusztig coefficients $\mu(y, w)$ are bounded on $c_0 \times c_0$. (Recall $\mu(w, y) = \mu(y, w)$ if $y \leq w$. Also, we set $\mu(y, w) = \mu(w, y) = 0$ if $y \nleq w$ and $w \nleq y$.)*

Proof. Let $y, w \in c_0$ and assume that $\mu(y,w) \neq 0$. Then by 1.4(a) we have $y \underset{L}{\sim} w$ or $y \underset{R}{\sim} w$. It is no harm to assume that $y \underset{L}{\sim} w$. Thus we can find $v \in W_0$ and $y', w' \in W$ such that $y = y'w_0d_v^{-1}$ and $w = w'w_0d_v^{-1}$ and $l(y) = l(y') + l(w_0) + l(d_v^{-1})$ and $l(w) = l(w') + l(w_0) + l(d_v^{-1})$. Using 2.2(c) we can see that $\mu(y,w) = \mu(y'w_0, w'w_0)$. Thus, to prove the theorem we only need to show that μ is bounded on $\Gamma_0 \times \Gamma_0$. When W has type $\tilde{A}_1$, it is easy to see that $\mu(y,w) \leq 1$ for all y, w in W. In general let $y = d_uxw_0,\ w = d_{u'}x'w_0 \in \Gamma_0$, where $u, u' \in W_0$ and $x, x' \in X^+$, be such that $\mu(y,w) \neq 0$. If $y \underset{R}{\sim} w$, then $u = u'$ and $x \leq x'$. This implies that $x'x^{-1}$ is in the root lattice and $l(y) - l(w) \equiv 0 (\mathrm{mod}\ 2)$ since $y \underset{L}{\sim} w$. This is impossible, so $u \neq u'$ and y and w are not in the same right cell. Thus $\gamma_{y^{-1},w,z} = 0$ for any z in W. By Springer formula in 1.3, we have $\mu(y,w) = \delta_{y^{-1},w,w_0}$. Set $x^* = w_0x^{-1}w_0 \in X^+$. By 2.2 (a), $C_{y^{-1}}C_w = S_{x^*}S_{x'}C_{w_0d_u^{-1}}C_{d_{u'}w_0}$. There are only finitely many z in W such that $h_{w_0d_u^{-1},d_{u'}w_0,z} \neq 0$ and if $h_{w_0d_u^{-1},d_{u'}w_0,z} \neq 0$ then $z = z_1w_0$ for some $z_1 \in X^+$. Thus we have $\mu(y,w) = \sum_{z_1 \in X^+} m_{x^*,x',w_0z_1^{-1}w_0}\delta_{w_0d_u^{-1},d_{u'}w_0,z_1w_0}$. Let $z_1^* = w_0z_1^{-1}w_0$. Then $m_{x^*,x',w_0z_1^{-1}w_0} = \dim \mathrm{Hom}_G(V(x^*) \otimes V(x'), V(z_1^*)) = \dim \mathrm{Hom}_G(V(z_1) \otimes V(x'), V(x)) \leq \dim V(z_1)$. Let

$$B = \max\{\sum_{z_1 \in X^+} \dim V(z_1)\delta_{w_0d_u^{-1},d_{u'}w_0,z_1w_0} \mid u, u' \in W_0\}.$$

Then we have $\mu(y,w) \leq B$. The theorem is proved. $\square$

Remark: When both y and w are in the left cell Γ_0, the theorem is proved in [**CPS2**, §7] by using representation theory. Keeping the same assumption $y, w \in \Gamma_0$, a referee pointed out that another proof of Theorem 3.1 can be obtained by observing that the coefficients $\mu(y,w)$ are also the “leading” coefficients of the “inverse” Kazhdan-Lusztig polynomials. Generically there are only finitely many of these (according to [**L1**, Corollary 11.9]) and the non-generic ones are obtained by taking alternating sums (see e.g. [**K**, Theorem 2.2]).

3.2. For the rest of this section we assume that $G = SL_{n+1}(\mathbf{C})$ and W is the corresponding extended affine Weyl group.

We number the simple roots $\alpha_1, \alpha_2, ..., \alpha_n$ and the simple reflections $s_0, s_1, ..., s_n$ of W as usual. Denote by $x_1, ..., x_n$ the fundamental weights. Let $\omega \in W$ be such that $\omega s_i \omega^{-1} = s_{i+1}$ (we set $s_{n+1} = s_0$). Then $x_1 = \omega^n s_2 \cdots s_n s_0$ and $x_n = \omega s_{n-1} s_{n-2} \cdots s_1 s_0$ and $x_1 x_n = s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_1 s_0$. We also have

$$x_2 = \omega^{n-1} s_4 s_5 \cdots s_n s_0 s_1 s_3 s_4 \cdots s_n s_0$$

and

$$x_{n-1} = \omega^2 s_{n-3} s_{n-2} s_{n-4} s_{n-3} \cdots s_0 s_1 s_n s_0.$$

Theorem 3.3. *Let $G = SL_{n+1}(\mathbf{C})$ and let W be the corresponding extended affine Weyl group. Let*

$$v = s_1 s_n s_0 s_2 s_3 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_2 s_1 s_n s_0$$

and let $x = \prod_{i=1}^n x_i^{a_i} \in X^+$ be a dominant weight such that all $a_i \geq 2$. Then $\mu(xw_0, vxw_0) = n + 2$ when $n \geq 4$.

Proof. From 3.2 we see that $y = x_1 x_n w_0 \leq v w_0$ and $l(vw_0) = l(y) + 1$. So we have $\mu(y, vw_0) = 1$. Let

$$w = s_{n-1} s_{n-2} \cdots s_2 s_1 s_n s_{n-1} \cdots s_2 s_4 s_5 \cdots s_n s_3 s_4 \cdots s_{n-1} s_1 s_n.$$

Then $R(w) = \{s_1, s_2, s_{n-1}, s_n\}$ and $w x_1 x_2 x_{n-1} x_n = s_0 v = d_w$. Thus we have $v = s_0 d_w \leq d_w$. So $v w_0 v^{-1}$ is a distinguished involution (see 2.1). Noting that $a(c_0) = l(w_0)$ and $l(v) = l(x_1 x_n) + 1$, we see that if $\delta_{w_0, vw_0, xw_0} \neq 0$ for an $x \in X^+$, then $l(x) + l(w_0) + a(w_0) - 1 \leq l(w_0) + l(vw_0)$. So $l(x) \leq l(v) + 1 = 2n + 2$. Noting that x is in the root lattice, we must have $x = x_1 x_n$ or $x = e$ (the neutral element). By 1.3, 2.2 (a) and 2.2 (b), we know that $\delta_{w_0, vw_0, x_1 x_n w_0} = 1$. A direct computation shows that $\delta_{w_0, vw_0, w_0} = 2$ (see Lemma 3.4 below). Now let $x \in X^+$, then we have $\mu(xw_0, vxw_0) = \sum_{z_1 \in X^+} m_{x^*, x, w_0 z_1^{-1} w_0} \delta_{w_0, vw_0, z_1 w_0}$ $= m_{x^*, x, x_1 x_n} + 2 = m_{x_1 x_n, x, x} + 2$. When $x = \prod_{i=1}^n x_i^{a_i}$ with all $a_i \geq 2$, we have $m_{x_1 x_n, x, x'} = 0$ if $x' x^{-1}$ is not a weight of $V(x_1 x_n)$, and $m_{x_1 x_n, x, x\lambda} = \dim V(x_1 x_n)_\lambda$, the dimension of the λ-weight space of $V(x_1 x_n)$. In particular, when $\lambda = e$, the neutral element, we get $m_{x_1 x_n, x, x} = \dim V(x_1 x_n)_e = n$, the dimension of the maximal torus of the Lie algebra $sl_{n+1}(\mathbf{C})$. The theorem is proved. $\square$

Lemma 3.4. *Let v be as in Theorem 3.3. If $n \geq 4$ then*

$$\mu(w_0, vw_0) = \delta_{w_0,vw_0,w_0} = 2.$$

PROOF. Note that c_0 is the lowest two-sided cell and Γ_0 is a left cell in c_0. This implies that for $y, w \in \Gamma_0$ with $y \leq w$, we have

$$P_{y,w} = q^a P_{sy,sw} + q^{1-a} P_{y,sw} - \sum_{\substack{z \in \Gamma_0 \\ y \leq z < sw \\ sz < z}} \mu(z, sw) q^{\frac{1}{2}(l(w)-l(z))} P_{y,z},$$

where $s \in S$ such that $sw \leq w$, $a = 0$ if $sy \leq y$ and $a = 1$ if $sy \geq y$.

We apply this formula to compute $\mu(w_0, vw_0)$.

Let $v_1 = s_1 v$. Note that $P_{sy,w} = P_{y,w}$ if $sw \leq w$ for $s \in S$. We have

$$P_{w_0,vw_0} = (1+q) P_{w_0,v_1w_0} - \sum_{\substack{z \in \Gamma_0 \\ w_0 \leq z < v_1 w_0 \\ s_1 z < z}} \mu(z, v_1 w_0) q^{\frac{1}{2}(l(vw_0)-l(z))} P_{w_0,z}.$$

We claim that

(1) If $z \in \Gamma_0,\ w_0 \leq z < v_1 w_0$ and $s_1 z < z$, then the degree of the polynomial $\mu(z, v_1 w_0) q^{\frac{1}{2}(l(vw_0)-l(z))} P_{w_0,z}$ is less than $\frac{1}{2}(l(vw_0) - l(w_0) - 1) = n$.

Note that $v_1 = s_n s_0 s_{n-1} s_{n-2} \cdots s_2 s_1 s_3 s_4 \cdots s_n s_0$. Thus if $z \in \Gamma_0$, $w_0 \leq z < v_1 w_0$, then z is one of the following elements:

$w_{ij} = s_n s_0 s_i s_{i-1} \cdots s_2 s_1 s_j s_{j+1} \cdots s_n s_0 w_0, \quad 1 \leq i \leq n-1,\ 3 \leq j \leq n;$

$u_{ij} = s_0 s_i s_{i-1} \cdots s_2 s_1 s_j s_{j+1} \cdots s_n s_0 w_0, \quad 1 \leq i \leq n-1,\ 3 \leq j \leq n;$

$v_{ij} = s_i s_{i-1} \cdots s_2 s_1 s_j s_{j+1} \cdots s_n s_0 w_0, \quad 1 \leq i \leq n-1,\ 3 \leq j \leq n;$

$w_i = s_i s_{i-1} \cdots s_2 s_1 s_0 w_0, \quad 1 \leq i \leq n-1;$

$u_j = s_j s_{j+1} \cdots s_n s_0 w_0, \quad 2 \leq j \leq n;$

$s_0 w_0,\ w_0.$

Note that $s_i v_1 w_0 \leq v_1 w_0$ for $i = 2, 3, ..., n-2$. If $i \geq 2$, then $s_1 w_{ij} \geq w_{ij}$. If $n = 1$, then $s_2 w_{ij} \leq w_{ij}$, thus $\mu(w_{ij}, v_1 w_0) = 0$. (We also have: when $n-2 \geq i$ and $j = n$, then w_{ij} is not in Γ_0; if $i \leq n-3$, then $s_{i+1} w_{ij} \geq w_{ij}$, so $\mu(w_{ij}, v_1 w_0) = 0$.)

Now we consider v_{ij}. When $i \leq n-3$, we have $s_{i+1} v_{ij} \geq v_{ij}$, so $\mu(v_{ij}, v_1 w_0) = 0$. If $i = n-2$ and $4 \leq j \leq n-1$, then $s_{j-2} v_{ij} \geq v_{ij}$, so $\mu(v_{ij}, v_1 w_0) = 0$. If $i = n-2$ and $j = 3$, then $s_1 v_{ij} \geq v_{ij}$. If $i = n-2$ and $j = n$, then the degree of $P_{w_0,v_{ij}}$ is 1, less than $\frac{1}{2}(l(v_{ij} - l(w_0) - 1) = \frac{n-1}{2}$

since $n \geq 4$. If $i = n-1$ and $4 \leq j \leq n$, then $s_{j-2}v_{ij} \geq v_{ij}$, so $\mu(v_{ij}, v_1w_0) = 0$. If $i = n-1$ and $j = 3$, then $s_1v_{ij} \geq v_{ij}$.

We have $\mu(w_0, u_{ij}) = 0$ since $s_0u_{ij} \leq u_{ij}$ and $s_0w_0 \geq w_0$ and $l(u_{ij}) - l(w_0) > 1$. We have $\mu(w_i, v_1w_0) = 0$ since $s_nv_1w_0 \leq v_1w_0$ and $s_nw_i \geq w_i$ and $l(v_1w_0) - l(w_i) > 1$. Note that $s_1u_j \geq u_j$ and $s_1s_0w_0 \geq s_0w_0$. Also we have $\mu(w_0, v_1w_0) = 0$ since $l(v_1)$ is even.

Thus we have shown that statement (1) is true.

Let $v_2 = s_nv_1$. Note that $P_{w_0,v_2w_0} = P_{s_0w_0,v_2w_0}$. We have

$$P_{w_0,v_1w_0} = (1+q)P_{s_0w_0,v_2w_0} - \sum_{\substack{z\in\Gamma_0 \\ w_0\leq z<v_2w_0 \\ s_nz<z}} \mu(z, v_2w_0)q^{\frac{1}{2}(l(v_1w_0)-l(z))}P_{w_0,z}.$$

Using a similar argument for (1) we see that

(2) If $z \in \Gamma_0$, $w_0 \leq z < v_2w_0$ and $s_nz < z$, then the degree of the polynomial $\mu(z, v_2w_0)q^{\frac{1}{2}(l(v_1w_0)-l(z))}P_{w_0,z}$ is less than $\frac{1}{2}(l(v_1w_0) - l(w_0) - 2) = n-1$.

Let $z_i = s_0s_is_{i-1}\cdots s_1s_3s_4\cdots s_ns_0w_0$. By a direct computation, we get

(3) $P_{s_0w_0,z_1} = P_{s_1s_0w_0,z_1} = 1+q$ and $P_{s_0w_0,z_2} = 1+2q+q^2$.

Let $u_2 = s_2s_3\cdots s_ns_0w_0$, by a direct computation we get

(4) $P_{u_2,z_i} = 1+q$ if $i \geq 3$ and $P_{u_2,z_2} = 1$.

Using this it is not difficult to get the following formula.

(5) $P_{s_0w_0,z_i} = (1+q)P_{s_0w_0,z_{i-1}} - qP_{s_0w_0,z_{i-2}} - \mu(s_0w_0, z_{i-1})q^{\frac{1}{2}(n+i-1)} - \mu(u_2, z_{i-1})q^{\frac{1}{2}i}$.

Thus we have

$$P_{s_0w_0,z_i} = \begin{cases} q^i + 2q^{i-1} + \text{lower degree terms} & \text{if } n > i+1 \\ 2q^{i-1} + \text{lower degree terms} & \text{if } n = i+1. \end{cases}$$

Note that $v_2w_0 = z_{n-1}$. Thus, we have $P_{s_0w_0,v_2} = 2q^{n-2}$+lower degree terms. Now using (1) and (2) we see that the lemma is true. $\square$

4. Some consequences

In this section we shall assume that G is simply connected and simple. We shall write the operation of X additively. For $x \in X$, denote by t_x the translation $y \to y + x$ of X. Let α_0 be the highest

short root of R. Set $s_0 = s_{\alpha_0} t_{p\alpha_0}$ (recall that we use s_α for the reflection on $E = X \otimes \mathbf{R}$ corresponding to $\alpha \in R$). Let W' be the subgroup of $GL(E)$ generated by all s_α ($\alpha \in R$) and s_0. Then W' is an affine Weyl group and is isomorphic to the group $W_0 \ltimes pQ$, where Q stands for the root lattice. Here p could be any positive (or negative) integer, but it is convenient here to suppose p is a fixed prime.

We shall further identify W' with the affine Weyl group W defined in [**L1**, section 1.1] and let W' act on X through the affine Weyl group defined in loc.cit. We denote this action by $*$. Then in terms of this new action, for $w \in W'$, we have $w * (-\rho) = w^{-1}(-\rho)$, and w is in Γ_0 if and only if $w * (-\rho) - \rho$ is dominant, where ρ is the sum of all fundamental weights. Note that $w * (-\rho) = w^{-1}(-\rho)$ is just a fact; it does not imply that $w * (u * (-\rho)) = w^{-1}(u * (-\rho))$ for w, u in W'.

Now let $G = SL_{n+1}(\mathbf{C})$ and we number the simple reflections of W_0 as usual. Let v be as in Theorem 3.3 and $\beta = \alpha_2 + \cdots + \alpha_{n-1}$. Then we have $v = s_\beta t_{p\varpi_2 + p\varpi_{n-1}}$ (we use ϖ_i for the fundamental weight corresponding to the simple root α_i). Let $\lambda = t_{2p\rho} w_0 * (-\rho) - \rho = 2p\rho$ and $\mu = v t_{2p\rho} w_0 * (-\rho) - \rho$. We have $\mu = w_0 t_{-2p\rho} t_{-p\varpi_2 - p\varpi_{n-1}} s_\beta(-\rho) - \rho = 2p\rho + (n-2)(\varpi_1 + \varpi_n) + (p-n+2)(\varpi_2 + \varpi_{n-1})$. We have $\langle \lambda + \rho, \alpha_0^\vee \rangle = 2pn + n$ and $\langle \mu + \rho, \alpha_0^\vee \rangle = 2pn + 2p + n$. The Jantzen region is defined to be the set of all vectors ν with $0 \leq \langle \nu + \rho, \alpha_0^\vee \rangle \leq p(p-h+2)$, where h is the Coxeter number. For $G = SL_{n+1}(\mathbf{C})$, we know $h = n+1$. Thus if $p \geq 3n+2$, then $p(p-h+2) = p(p-n+1) \geq 2pn + 3p > 2pn + 2p + n$. Thus both λ and μ are in the Jantzen region.

Now replace $\mathbf{C}$ with an algebraically closed field k of characteristic p and let H be a simply connected and simple algebraic group over k. It is known for each root system that when p is sufficiently large, Lusztig's conjecture for modular representations of algebraic groups (Lusztig's modular conjecture in short, see [**L0**] for the formulation) is true for irreducible modules of H with highest weight in the Jantzen region.

For $H = SL_{n+1}(k)$, by Theorem 3.3 we know that the Kazhdan-Lusztig coefficient $\mu(t_{2p\rho} w_0, v t_{2p\rho} w_0)$ of $P_{t_{2p\rho} w_0, v t_{2p\rho} w_0}$ is $n+2$ if $n \geq 4$. It is known that the coefficient is related to the dimension of the first

extension groups for extensions between certain irreducible modules for H. See for example [**CPS2**] and references therein.[2] So we conclude that the dimension goes to infinity when n increases.

Remark: We do not know if the quantities $\mu(w_0, w)$ that correspond to 1-cohomology dimensions can go to infinity. We do have some plausible candidates, though. Let $\omega \in W_0 \ltimes pX$ be such that $\omega(s_i) = s_{i+1}$ for any $0 \leq i \leq n$ (set $s_{n+1} = s_0$) for any $0 \leq i \leq n$. Assume that $n = 2k-1$ is odd. Then $w = s_k\omega^k t_{-p\rho}$ is in $W' \cap \Gamma_0$, and the weight $w * (-\rho) - \rho$ is p-restricted. The explicit form of $w * (-\rho) - \rho$ is $(p-2)\rho - \alpha_0 - (p-n-1)\varpi_k$. For $n = 5$ and p large, this is the same weight which gave 1-cohomology dimensions of 3 with irreducible coefficients in [**S**] (found there by computer calculations of Kazhdan-Lusztig polynomials). For general odd $n > 1$, the weight is in the second top alcove of the fundamental p-box $\{v \in X \otimes \mathbf{R} \mid 0 < \langle v + \rho, \alpha_i^\vee \rangle < p,\ 1 \leq i \leq n\}$. If $n > 1$ is odd, then $l(t_{-p\rho}) - l(w_0)$ is even, so $\mu(w_0, t_{-p\rho}) = 0$. Thus $w * (-\rho) - \rho$ is the largest possible p-restricted weight in the orbit $W * (-\rho) - \rho$ whose corresponding irreducible module could have nonzero 1-cohomology. For $n = 4k \geq 4$, set $w = s_0 t_{-p\rho}$. Then $w * (-\rho) - \rho = (p-2)\rho - (p-n)\alpha_0$ is also the largest possible p-restricted weight whose corresponding irreducible module could have nonzero 1-cohomology. In fact, when $n = 4$, $w = s_0 t_{-p\rho}$ is just the vw_0 in Lemma 3.4. The question is whether $\mu(w_0, w)$ goes to infinity when k increases? (When $n = 4k+2$, we have not found a similar candidate.)

5. Representation-theoretic argument

The aim of this section is to prove a version of Theorem 3.3 by a representation-theoretic argument. The result is weaker, in that we do not exactly compute the relevant Kazhdan-Lusztig coefficents, but only get a good lower bound. However, the hypotheses required are also somewhat weaker. As in §4, we will assume p is sufficiently large so that the Lusztig modular conjecture holds for the group $G = SL_{n+1}(k)$,

[2] It is useful to note, in comparing notation, that $P_{y,w} = P_{y^{-1},w^{-1}}$, and so $\mu(y, w) = \mu(y^{-1}, w^{-1})$, for all $y < w$ in W. See [**KL**].

with k an algebraically closed field of characteristic p. We also require $p \geq 3n+2$, as in §4, at least to begin our discussion. This assumption was used in §4 to ensure the weights λ and μ there were in the Jantzen region. Eventually, in this section, we will be able to obtain a good lower bound for more weights, and we will only need that $p \geq 3n$ to enable some of them to be in the Jantzen region. This latter inequality, fortunately, also ensures that a formula of Andersen may be applied.

5.1. Andersen's Formula. For dominant weights $\lambda = \lambda_0 + p\lambda_1$ and $\mu = \mu_0 + p\mu_1$ with $\lambda_0 \neq \mu_0$ both restricted and λ_1, μ_1 both dominant, the formula asserts (for any connected semisimple group) that

Theorem 5.1. *Assume $p \geq 3h-3$ (which is $3n$ for type A_n). Then*

$$dim\ Ext^1_G(L(\lambda), L(\mu))$$

$$= \sum_{\nu} dim\ Ext^1_G(L(\lambda_0 + p\nu), L(\mu_0))\, dim\ Hom_G(L(\lambda_1), L(\nu) \otimes L(\mu_1)).$$

Here ν ranges over all dominant weights satisfying $\lambda_0 + p\nu \leq 2(p-1)\rho + w_0(\mu_0)$. For a more complete statement, see [**CPS2**, 7.8] or [**A2**].

5.2. A lower bound for some $\mathbf{Ext}^1$ dimensions. Recall that $\varpi_1, \ldots, \varpi_n$ are the fundamental weights (indexed corresponding to $\alpha_1, \ldots, \alpha_n$). Set $a = n-2$ and put $\mu_0 = a(\varpi_1 + \varpi_n) + (p-a)(\varpi_2 + \varpi_{n-1})$ (note that this is the dominant weight $vw_0 * (-\rho) - \rho$ corresponding to vw_0, see Section 4). Let μ_1 be any weight of the form $\sum_{i=1}^n a_i\varpi_i$ with each $a_i \geq 2$, and put $\mu = \mu_0 + p\mu_1$. (Thus, if μ_1 is 2ρ, this is the same μ as in §4.) We will apply Andersen's formula in 5.1 to μ and to $\lambda = p\mu_1$. The following result is a weak version of Theorem 3.3, but still strong enough to show $\dim \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \to \infty$ as $n \to \infty$.

Theorem 5.2. *Let λ, μ be as above, and assume $n + \sum_{i=1}^n a_i \leq p$ (which certainly holds if $\mu_1 = 2\rho$). Then*

$$dim\ Ext^1_G(L(\lambda), L(\mu)) \geq n.$$

Proof. First, we show $\mathrm{Ext}^1_G(L(\lambda_0 + p\alpha_0), L(\mu_0)) \neq 0$. Here λ_0 is the zero weight, and $\alpha_0 = \varpi_1 + \varpi_n$. In particular,

$$L(\lambda_0 + p\alpha_0) \cong L(p\alpha_0),$$

and

$$\mathrm{Ext}^1_G(L(\lambda_0 + p\alpha_0), L(\mu_0)) \cong \mathrm{Ext}^1_G(L(p\alpha_0), L(\mu_0)).$$

Observe that the weight μ_0 is obtained from $p\alpha_0$ by reflection in the hyperplane

$$\{x \in \mathbb{R}^n |\ \langle x + \rho, (\alpha_2 + \ldots + \alpha_{n-1})^\vee \rangle = p\}.$$

We can also directly calculate the number $d(\mu_0)$ of hyperplanes of the form $\{x \in \mathbb{R}^n |\ \langle x + \rho, \alpha^\vee \rangle = mp\}$, with $\alpha > 0$ and $m \in \mathbb{Z}$, which separate μ_0 from 0, and compare $d(\mu_0)$ with $d(p\alpha_0)$. We find that μ_0 and $p\alpha_0$ are on opposite sides of only the hyperplanes defined by $\alpha = \alpha_1$, $m = 1$; $\alpha = \alpha_n$, $m = 1$; $\alpha = \alpha_0 - \alpha_1$, $m = 2$; $\alpha = \alpha_0 - \alpha_n$, $m = 2$; $\alpha = \alpha_0 - \alpha_1 - \alpha_n$, $m = 1$. For the first two of these five hyperplanes, the weight μ_0 is on the same side as 0. For the last three, $p\alpha_0$ is on the same side as 0. Thus, $d(\mu_0) = d(p\alpha_0) + 1$. It now follows from [**J**, II, 6.24] that $L(p\alpha_0)$ is a composition factor of the costandard module $\nabla(\mu_0) = H^0(\mu_0)$. Also, the strong linkage principle implies that $p\alpha_0$ is maximal among the highest weights of composition factors of $\nabla(\mu_0)/L(\mu_0)$. Thus, there is a nonzero homomorphism $\Delta(p\alpha_0) \to \nabla(\mu_0)/L(\mu_0)$, and this shows $\mathrm{Ext}^1_G(\Delta(p\alpha_0), L(\mu_0)) \neq 0$. However, since p is large enough that the Lusztig conjecture holds, and both $p\alpha_0$ and μ_0 lie in the Jantzen region (we calculate this using $p > n$), we have

$$\mathrm{Ext}^1_G(L(p\alpha_0), L(\mu_0)) \cong \mathrm{Ext}^1_G(\Delta(p\alpha_0), L(\mu_0))$$

by a well-known result of Andersen [**A1**, Proposition 2.8]. (We take the λ, y, w, s there to be $-2\rho, t_{p\alpha_0}w_0, s_2vw_0, s_2$ respectively. Note that s_2vw_0, s_2 can be replaced by $s_{n-1}vw_0, s_{n-1}$ respectively.) This proves the claim.

To prove the theorem, it is now sufficient by Andersen's formula in 5.1, to show

$$\dim \mathrm{Hom}_G(L(\alpha_0), L(\mu_1) \otimes L(\mu_1)) \geq n.$$

Note that α_0 and μ_1 lie in the closure of the lowest p-alcove. (This uses our inequality $\sum_{i=1}^n a_i + n \leq p$.) Thus

$$L(\alpha_0) \cong \Delta(\alpha_0) \cong \nabla(\alpha_0), \quad L(\mu_1) \cong \Delta(\mu_1) \cong \nabla(\mu_1),$$

and

$$\begin{aligned} \mathrm{Hom}_G(L(\alpha_0), L(\mu_1) \otimes L(\mu_1)) &\cong \mathrm{Hom}_G(\Delta(\mu_1), \Delta(\alpha_0) \otimes \nabla(\mu_1)) \\ &\cong \mathrm{Hom}_G(\Delta(\mu_1), \mathrm{Ind}_B^G(\Delta(\alpha_0) \otimes k(\mu_1)), \end{aligned}$$

where B is the Borel subgroup associated to the negative roots, and $k(\mu_1)$ is the one-dimensional B module with weight μ_1. The module $\Delta(\alpha_0)$ is just the Lie algebra $\mathfrak{g}$ of G, with the usual adjoint action. The Borel subalgebra $\mathfrak{b}$ with negative root spaces is a B-submodule, and $\mathfrak{b} \otimes k(\mu_1)$ is a B-submodule of $\mathfrak{g} \otimes k(\mu_1) = \Delta(\alpha_0) \otimes k(\mu_1)$. This B-submodule has a B-quotient which is a direct sum of n copies of $k(\mu_1)$ and all the other composition factors have the form $k(\mu_1 - \beta)$, where β is a positive root. Because of our assumption that all a_i are at least 2, the weights $\mu_1 - \beta$ are all dominant. (It is possible to carry through a version of the argument which follows with a requirement weaker than "dominant", so it is actually enough that each a_i be at least 1. See 5.3(c) below.) Applying Kempf's vanishing theorem, we find that $\mathrm{Ind}_B^G(\mathfrak{b} \otimes k(\mu_1))$ has a quotient isomorphic to a direct sum of n copies of $\nabla(\mu_1)$. The kernel of the map to this quotient is filtered by modules $\nabla(\mu_1 - \beta)$. Since $\nabla(\mu_1) \cong \Delta(\mu_1)$, and there are no nontrivial extensions of a standard module by a costandard module, the quotient map is split. Since $\mathrm{Ind}_B^G(\mathfrak{b} \times k(\mu_1)) \subseteq \mathrm{Ind}_B^G(\Delta(\alpha_0) \otimes k(\mu_1))$, we have

$$\begin{aligned} &\dim \mathrm{Hom}_G(\Delta(\mu_1), \mathrm{Ind}_B^G(\Delta(\alpha_0) \otimes k(\mu_1)) \\ &\quad \geq \dim \mathrm{Hom}_G(\Delta(\mu_1), \mathrm{Ind}_B^G(\mathfrak{b} \otimes k(\mu_1))) \geq n. \end{aligned}$$

Taken with the isomorphisms and discussions above, this completes the proof of the theorem. □

5.3. Remarks. (a) The assumption $n + \sum_{i=1}^n a_i \leq p$ is used only to guarantee $L(\mu_1) \cong \Delta(\mu_1) \cong \nabla(\mu_1)$. In [**CPS2**, §7] it is suggested that Andersen's formula should be true, appropriately formulated (and with a similar proof) for quantum enveloping algebras at a root of

unity. In such a formulation, the terms involving Hom_G would instead involve a Hom over the ordinary characteristic 0 enveloping algebra of $\mathfrak{g}$. Thus, the required isomorphisms on $L(\mu_1)$ would hold without the assumed inequality. That is, Theorem 5.2 should hold at a p^{th} root of unity without the assumed inequality (if $p > n$). Essentially, use of the quantum group frees the representation theory from dependence on the Jantzen region and allows a 1–1 correspondence between affine Weyl group results and representation theory results in our context.

(b) If Lemma 3.4 is interpreted as an Ext^1 result and fed into Andersen's formula, it enables n in the inequality in Theorem 5.2 to be replaced with $n+2$. This result is almost as good as Theorem 3.3, though the latter gives the resulting inequality as an equality.

(c) The hypotheses $a_i \geq 2$ ($i = 1, \ldots, n$) in 5.2 (and in the auxiliary remarks 5.3(a), 5.3(b) above) *can be weakened to just assuming* $a_i \geq 1$ ($i = 1, \ldots, n$). To see this, note that $a_i \geq 2$ condition was used only to guarantee that the weights $\mu_1 - \beta$ were all dominant, with β any positive root. The dominance guaranteed, through Kempf's theorem, that the higher derived functors $R^1\mathrm{Ind}_B^G(k(\mu_1 - \beta))$ were zero. However, this is true also when $\langle \mu_1 - \beta, \alpha_i^\vee \rangle = -1$ for some i. (All $R^j\mathrm{Ind}_B^G(k(\mu_1 - \beta))$ vanish in this case; see [**J**, II,5.4(a)].) We find, with the assumption $a_i \geq 1$, for $1 \leq i \leq n$, that the kernel of the map from $\mathrm{Ind}_B^G(\mathfrak{b} \otimes k(\mu_1))$ onto a direct sum of n copies of $k(\mu_1)$, is again filtered by costandard modules. (Any potential section $\mathrm{Ind}_B^G k(\mu_1 - \beta)$, in which $\mu_1 - \beta$ is not dominant, is just zero.)

Thus, assuming that Andersen's formula extends to the quantum case, as discussed in 5.3(a), we have, in the notation of Theorem 3.3,

$$\mu(xw_0, vxw_0) \geq n + 2,$$

assuming, as in Theorem 3.3, that $n \geq 4$, but weakening the requirement $a_i \geq 2$ to $a_i \geq 1$, for all $i = 1, 2, \ldots, n$. The assumption $p \geq 3n+2$ can be replaced, then, with $p \geq 3n$. (Still p must be large enough for the Lusztig conjecture to hold.)

(d) Andersen observed that the inequality in Theorem 5.2 can be improved to an equality. More precisely, for all dominant weights μ_1,

if $p > n+1$ and p is large enough that the Lusztig modular conjecture holds for G, one has

$$\dim \mathrm{Ext}^1_G(\Delta(\mu_1)^{(1)}, L(\mu_0) \otimes \nabla(\mu_1)^{(1)}) = n + 2 - f(\mu_1),$$

where $f(\mu_1)$ is the number of simple roots orthogonal to μ_1. To see this one first notes that $H^1(G_1, H^0(\mu_0)) = 0$ ([**AJ**]), where G_1 is the first Frobenius kernel of G. Here the cohomology group $H^0(\mu_0)$ is the G-module $\nabla(\mu_0)$. Then

$$H^1(G_1, L(\mu_0)) = H^0(G_1, H^0(\mu_0)/L(\mu_0)) = L(\alpha_0) \oplus k \oplus k.$$

Here the first summand comes as in 5.2, and the appearance of two copies of k is a consequence of Lemma 3.4 (which gives $H^1(G, L(\mu_0)) = k \oplus k$). Since $M = L(\alpha_0) \otimes \nabla(\mu_1)$ has a good filtration $M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{r-1} \supset M_r = 0$ (this uses $L(\alpha_0) = \nabla(\alpha_0)$, which holds for p prime to $n+1$), the dimension of $\mathrm{Hom}_G(\Delta(\mu_1), M)$ is equal to the number of occurrences of $\nabla(\mu_1)$ in subquotients M_0/M_1, M_1/M_2, ..., M_{r-1}/M_r of any good filtration of M, i.e. $n - f(\mu_1)$.

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